Lambda Calculus and Types

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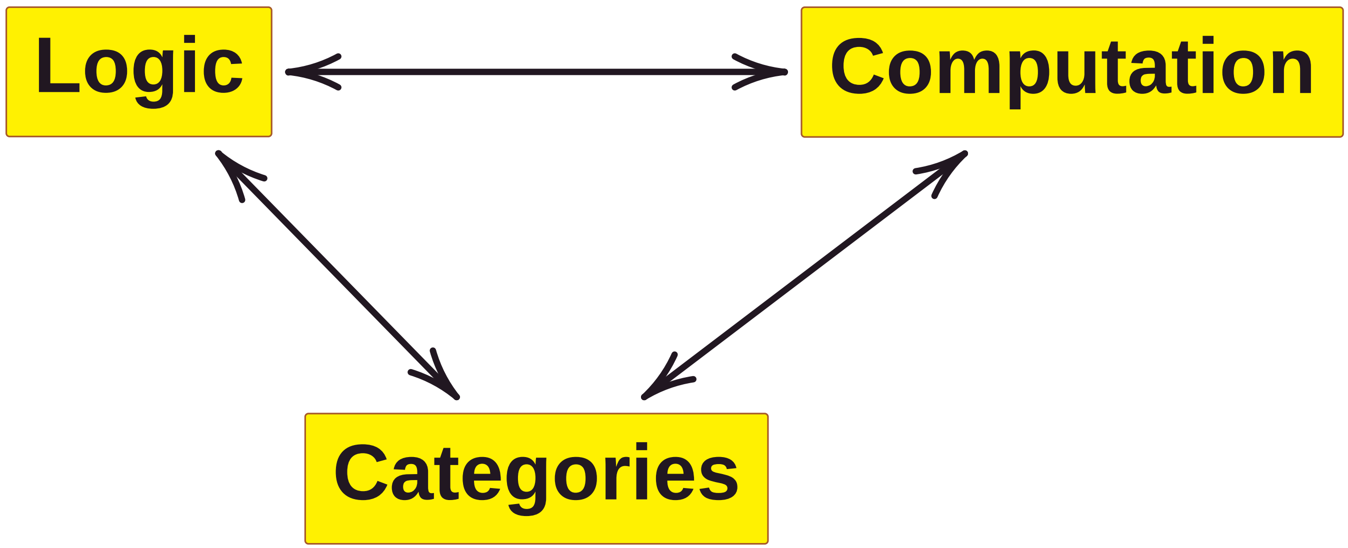
Lecture 5: The Curry-Howard Correspondence



- It is the upper link (Logic–Computation) that is usually attributed to H. B. Curry and W. A. Howard, although the idea is related to the operational interpretation of intuitionistic logic given in various formulations by Brouwer, Heyting and Kolmogorov.
- The link to Categories is mainly due to the pioneering work of J. Lambek in the 1970's.

Logic and Formal Proofs

- Traditional introductions to logic focus on Hilbert-style proof systems: generating the set of *theorems* of a system from a set of axioms by applying rules of inference (e.g. Modus Ponens).

- A key step in logic took place in the 1930's, with the advent of *Gentzen-style systems*. Instead of focusing on theorems, look more generally and symmetrically at proofs:

  What follows from what

- The idea is to study the *space of formal proofs* as a mathematical structure in its own right, rather than to focus only on

  $$\text{Provability} \longleftrightarrow \text{Truth}$$

  (i.e. the usual notions of soundness and completeness).

A logical formula A can be ***proved from assumptions*** $A_1, \ldots, A_n$:

$$A_1, \cdots, A_n \vdash A$$

We call the above a ***sequent***. We use $\Gamma, \Delta, \ldots$ to range over finite sets of formulas, writing Γ, A for $\Gamma \cup \{A\}$, etc.

Proofs of sequents are built using the following rules.

$$\frac{}{\Gamma, A \vdash A}\ \mathrm{Id}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-intro} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-elim}_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-elim}_2$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \supset\text{-intro} \qquad \frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \supset\text{-elim}$$

Simply-Typed λ-calculus with products

The system $\boldsymbol{\lambda}^{\mathbb{A}}_{\to\times}$ extends $\boldsymbol{\lambda}^{\mathbb{A}}_{\to}$ with ***products***:

$$\frac{}{a \in \textbf{Typ}}\ a \in \mathbb{A} \qquad \frac{A \in \textbf{Typ} \quad B \in \textbf{Typ}}{(A \to B) \in \textbf{Typ}} \qquad \frac{A \in \textbf{Typ} \quad B \in \textbf{Typ}}{(A \times B) \in \textbf{Typ}}$$

The set of ***terms*** Λ will now be given by the rules:

$$\frac{}{x \in \Lambda}\ x \in \mathcal{V} \qquad \frac{s \in \Lambda \quad t \in \Lambda}{(st) \in \Lambda} \qquad \frac{s \in \Lambda}{(\lambda x.s) \in \Lambda}\ x \in \mathcal{V}$$

$$\frac{s \in \Lambda \quad t \in \Lambda}{\langle s, t \rangle \in \Lambda} \qquad \frac{s \in \Lambda}{(\pi_1 s) \in \Lambda} \qquad \frac{s \in \Lambda}{(\pi_2 s) \in \Lambda}$$

We use (and extend) our common parenthesis-saving techniques.

Variable

$$\frac{}{\Gamma, x : A \vdash x : A}$$

Product

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \langle t, u \rangle : A \times B} \qquad \frac{\Gamma \vdash v : A \times B}{\Gamma \vdash \pi_1 v : A} \qquad \frac{\Gamma \vdash v : A \times B}{\Gamma \vdash \pi_2 v : B}$$

Function

$$\frac{\Gamma, x : B \vdash t : A}{\Gamma \vdash \lambda x.\, t : B \to A} \qquad \frac{\Gamma \vdash t : B \to A \qquad \Gamma \vdash u : B}{\Gamma \vdash tu : A}$$

Reduction rules

Beta reduction is now built from the basic rules:

$$
\begin{array}{lll}
(\lambda x.\, t)u & \to_\beta & t[u/x] \\
\pi_1\langle t, u\rangle & \to_\beta & t \\
\pi_2\langle t, u\rangle & \to_\beta & u
\end{array}
$$

while η-reduction from:

$$
\begin{array}{llll}
\lambda x.\, tx & \to_\eta & t & x \text{ not free in } t\text{, at function types} \\
\langle \pi_1 v, \pi_2 v\rangle & \to_\eta & v & \text{at product types}
\end{array}
$$

Simple Type System for $\times$, $\rightarrow$	**Natural Deduction System for** $\wedge$, $\supset$
$\dfrac{}{\Gamma, x : A \vdash x : A}$	$\dfrac{}{\Gamma, A \vdash A}\ \mathrm{Id}$
$\dfrac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \langle t, u \rangle : A \times B}$	$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\mathrm{I}$
$\dfrac{\Gamma \vdash v : A \times B}{\Gamma \vdash \pi_1 v : A}$	$\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\mathrm{E}_1$
$\dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \rightarrow B}$	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \supset\mathrm{I}$
$\dfrac{\Gamma \vdash t : A \rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}$	$\dfrac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \supset\mathrm{E}$

The Curry-Howard Correspondence

If we equate

$$\begin{array}{ccc} \wedge & \equiv & \times \\ \supset & \equiv & \rightarrow \end{array}$$

they are the same!

In fact, the correspondence works on three levels:

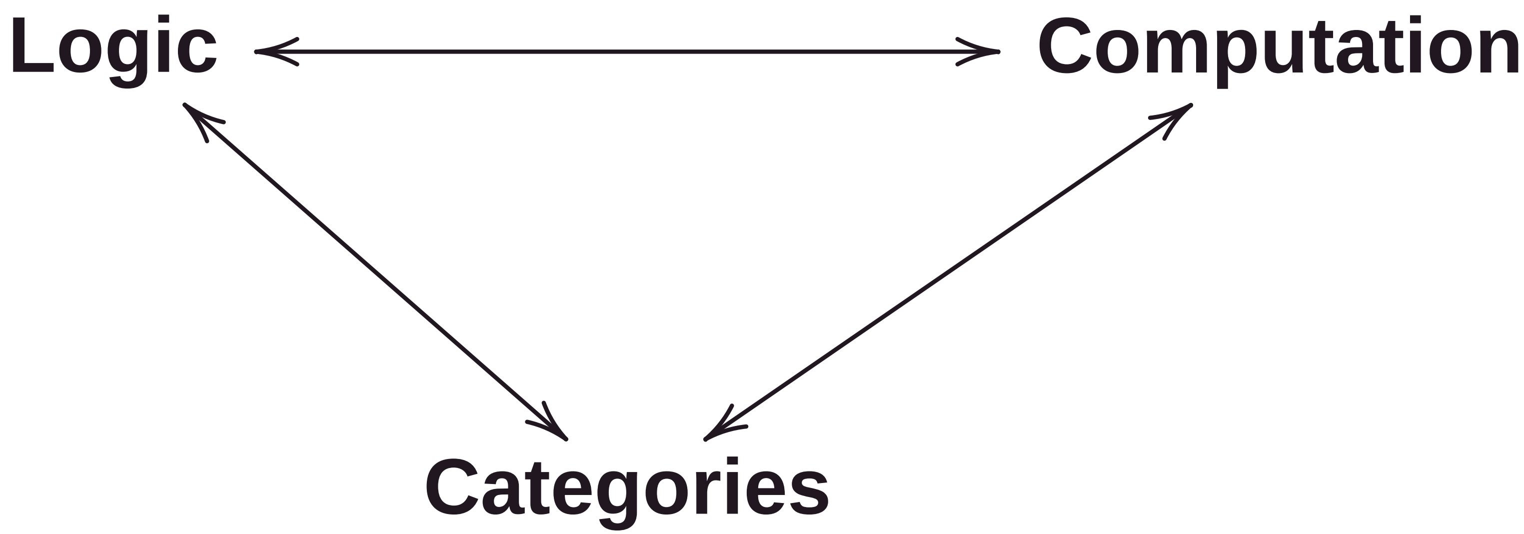
Formulas	$\equiv$	Types
Proofs	$\equiv$	Terms
Proof transformations	$\equiv$	Term reductions

Reduction/transformation results in a **normal form**; a derivation/proof in which all redexes/lemmas have been eliminated, resulting in an **explicit but much longer expression**.

Even simply typed lambda calculus has enormous (*non-elementary*) complexity.

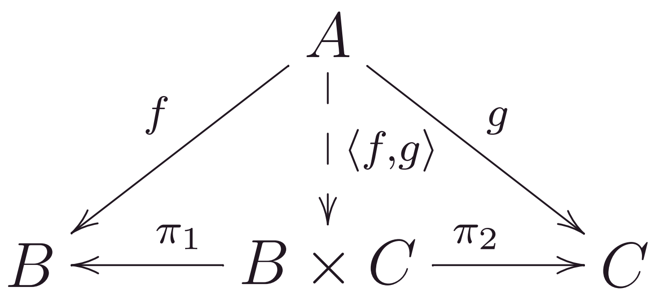
The Connection to Categories

That was the link between Logic and Computation. We now complete the triangle by showing the connection to Categories.



Let us recall the ingredients of a cartesian closed category $\mathcal{C}$:

- **Finite products.** Triples $B \xleftarrow{\pi_1} B \times C \xrightarrow{\pi_2} C$ such that:

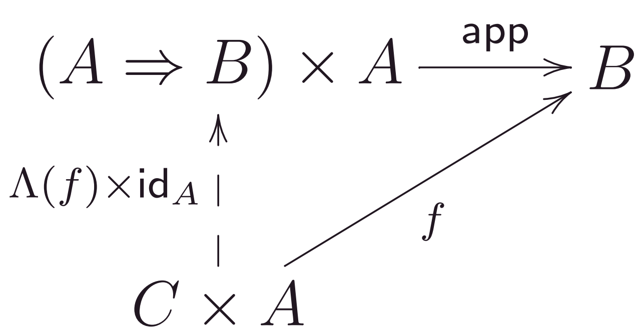


$$\pi_1 \circ \langle f, g \rangle = f \qquad \pi_2 \circ \langle f, g \rangle = g \qquad (\beta)$$

$$\forall h : A \to B \times C.\ h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle \quad (\eta)$$

  and an object $\mathbf{1}$ with unique morphisms $\tau_A : A \longrightarrow \mathbf{1}$.

- **Exponentials.** Pairs $\mathsf{app} : (A \Rightarrow B) \times A \to B$ such that:



$$\mathsf{app} \circ (\Lambda(f) \times \mathsf{id}_A) = f \qquad (\beta)$$

$$\forall h : C \to (A \Rightarrow B).$$

$$h = \Lambda(\mathsf{app} \circ (h \times \mathsf{id}_A)) \quad (\eta)$$

  where $f_1 \times f_2 = \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle$.

Connection to Computation

Let $\mathcal{C}$ be a CCC. We translate the simply-typed λ-calculus into $\mathcal{C}$, so that to each typed term $x_1 : A_1, ..., x_k : A_k \vdash t : A$ corresponds an arrow:

$$[\![t]\!] : [\![A_1]\!] \times \cdots \times [\![A_k]\!] \longrightarrow [\![A]\!]$$

We assume $\mathcal{C}$ contains objects $\tilde{a}$, one for each atomic type a.

The semantic translation

- $[\![a]\!] := \tilde{a} \quad , \quad [\![A \to B]\!] := [\![A]\!] \Rightarrow [\![B]\!]$
- $[\![x_1 : A_1, \cdots, x_n : A_n \vdash x_i : A_i]\!] := \pi_i : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \longrightarrow [\![A_i]\!]$
- $$\frac{[\![\Gamma \vdash t : A \times B]\!] = f : [\![\Gamma]\!] \longrightarrow [\![A]\!] \times [\![B]\!]}{[\![\Gamma \vdash \pi_1 t : A]\!] := [\![\Gamma]\!] \xrightarrow{f} [\![A]\!] \times [\![B]\!] \xrightarrow{\pi_1} [\![A]\!]}$$
- $$\frac{[\![\Gamma \vdash t : A]\!] = f : [\![\Gamma]\!] \longrightarrow [\![A]\!] \qquad [\![\Gamma \vdash u : B]\!] = g : [\![\Gamma]\!] \longrightarrow [\![B]\!]}{[\![\Gamma \vdash \langle t, u \rangle : A \times B]\!] := [\![\Gamma]\!] \xrightarrow{\langle f, g \rangle} [\![A]\!] \times [\![B]\!]}$$
- $$\frac{[\![\Gamma, x : A \vdash t : B]\!] = f : [\![\Gamma]\!] \times [\![A]\!] \longrightarrow [\![B]\!]}{[\![\Gamma \vdash \lambda x.t : A \to B]\!] := \Lambda(f) : [\![\Gamma]\!] \longrightarrow ([\![A]\!] \Rightarrow [\![B]\!])}$$
- $$\frac{[\![\Gamma \vdash t : A \to B]\!] = f \qquad [\![\Gamma \vdash u : A]\!] = g}{[\![\Gamma \vdash tu : B]\!] := [\![\Gamma]\!] \xrightarrow{\langle f, g \rangle} ([\![A]\!] \Rightarrow [\![B]\!]) \times [\![A]\!] \xrightarrow{\mathsf{app}} [\![B]\!]}$$

Naturality of Pairing and Currying

Given $f_1 : D_1 \longrightarrow E_1$, $f_2 : D_2 \longrightarrow E_2$, recall:

$$f_1 \times f_2 = \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle : D_1 \times D_2 \longrightarrow E_1 \times E_2$$

Lemma. *The following hold, for all relevant morphisms.*

- $\langle h, k \rangle \circ f = \langle h \circ f, k \circ f \rangle$
- $(f \times g) \circ \langle h, k \rangle = \langle f \circ h, g \circ k \rangle$
- $\Lambda(f) \circ g = \Lambda(f \circ (g \times \mathsf{id}))$

Substitution Revisited

We consider *simultaneous substitution* for all the free variables in a term

$$x_1 : A_1, \ldots, x_k : A_k \vdash t : A$$

Given $\Gamma \vdash t_i : A_i$, $1 \leq i \leq k$, we define

$$t[\vec{t}/\vec{x}] \quad \equiv \quad t[t_1/x_1, \ldots, x_k/t_k]$$

$$\begin{array}{rcl} x_i[\vec{t}/\vec{x}] & := & t_i \\ tu[\vec{t}/\vec{x}] & := & (t[\vec{t}/\vec{x}])(u[\vec{t}/\vec{x}]) \\ \lambda x.t[\vec{t}/\vec{x}] & := & \lambda x.(t[\vec{t}, x/\vec{x}, x]) \end{array}$$

Lemma.

$$[\![t[t_1/x_1,\ldots,x_k/t_k]]\!] = [\![t]\!] \circ \langle [\![t_1]\!],\ldots,[\![t_k]\!]\rangle.$$

Proof. By induction on the structure of t.

(1) $t \equiv x_i$.

$$[\![x_i[\vec{t}/\vec{x}]]\!] = [\![t_i]\!] = \pi_i \circ \langle [\![t_1]\!],\ldots,[\![t_k]\!]\rangle = [\![x_i]\!] \circ \langle [\![t_1]\!],\ldots,[\![t_k]\!]\rangle.$$

(2) $t \equiv uv$. We abbreviate $\langle [\![t_1]\!],\ldots,[\![t_k]\!]\rangle$ to $\langle [\![\vec{t}]\!]\rangle$.

$$\begin{array}{rcll}
[\![(uv)[\vec{t}/\vec{x}]]\!] & = & [\![(u[\vec{t}/\vec{x}])(v[\vec{t}/\vec{x}])]\!] & \text{Defn of substitution} \\
 & = & \mathsf{app} \circ \langle [\![u[\vec{t}/\vec{x}]]\!], [\![v[\vec{t}/\vec{x}]]\!]\rangle & \text{Defn of semantic function} \\
 & = & \mathsf{app} \circ \langle [\![u]\!] \circ \langle [\![\vec{t}]\!]\rangle, [\![v]\!] \circ \langle [\![\vec{t}]\!]\rangle\rangle & \text{Induction hyp} \\
 & = & \mathsf{app} \circ \langle [\![u]\!], [\![v]\!]\rangle \circ \langle [\![\vec{t}]\!]\rangle & \text{Naturality} \\
 & = & [\![uv]\!] \circ \langle [\![\vec{t}]\!]\rangle & \text{Defn of semantic function}
\end{array}$$

(3) $t \equiv \lambda x.u$.

$$
\begin{array}{rcll}
[\![(\lambda x.u)[\vec{t}/\vec{x}]]\!] & = & [\![\lambda x.(u[\vec{t}, x/\vec{x}, x])]\!] & \text{Defn. of substitution} \\
& = & \Lambda([\![u[\vec{t}, x/\vec{x}, x]]\!]) & \text{Defn. of semantic function} \\
& = & \Lambda([\![u]\!] \circ \langle [\![\vec{t}x]\!] \rangle) & \text{Induction hyp.} \\
& = & \Lambda([\![u]\!] \circ \langle \langle [\![\vec{t}]\!] \rangle \circ \pi_1, \pi_2 \rangle) & \text{Unwrapping } \langle \cdots \rangle \\
& = & \Lambda([\![u]\!] \circ (\langle [\![\vec{t}]\!] \rangle \times \mathsf{id})) & \text{Naturality} \\
& = & \Lambda([\![u]\!]) \circ \langle [\![\vec{t}]\!] \rangle & \text{Naturality} \\
& = & [\![\lambda x.u]\!] \circ \langle [\![\vec{t}]\!] \rangle & \text{Defn. of semantic function}
\end{array}
$$

$\square$

Validating the conversion rules

We can now show that the conversion rules of the λ-calculus are *preserved* by the translation, and hence the interpretation is *sound*.

β-conversion: $(\lambda x.\, t)u =_\beta t[u/x]\,,\ \pi_1\langle t, u\rangle =_\beta t\,,\ \pi_2\langle t, u\rangle =_\beta u$

$$\begin{aligned}
[\![(\lambda x.t)u]\!] &= \mathsf{app} \circ \langle \Lambda([\![t]\!]), [\![u]\!]\rangle && \text{Defn. of semantics}\\
&= \mathsf{app} \circ (\Lambda([\![t]\!]) \times \mathsf{id}) \circ \langle \mathsf{id}_{[\![\Gamma]\!]}, [\![u]\!]\rangle && \text{Naturality}\\
&= [\![t]\!] \circ \langle \mathsf{id}_{[\![\Gamma]\!]}, [\![u]\!]\rangle && (\beta)\\
&= [\![t[\vec{x}, u/\vec{x}, x]]\!] && \text{Substitution lemma.}
\end{aligned}$$

$$[\![\pi_1\langle t, u\rangle]\!] = \pi_1 \circ [\![\langle t, u\rangle]\!] = \pi_1 \circ \langle [\![t]\!], [\![u]\!]\rangle \overset{(\beta)}{=} [\![t]\!]$$

η-conversion: $t =_\eta \lambda x.tx\,,\ \langle \pi_1 t, \pi_2 t\rangle =_\eta t$

$$\begin{aligned}
[\![\lambda x.tx]\!] &= \Lambda(\mathsf{app} \circ ([\![t]\!] \times \mathsf{id})) = [\![t]\!] && (\eta) \text{ for } \Rightarrow\\
[\![\langle \pi_1 t, \pi_2 t\rangle]\!] &= \langle \pi_1 \circ [\![t]\!], \pi_2 \circ [\![t]\!]\rangle = [\![t]\!] && (\eta) \text{ for } \times
\end{aligned}$$

Hence, since the semantic function is compositional,

$$s =_{\beta\eta} t \implies [\![s]\!] = [\![t]\!]$$

The morale is:

CCC's are the **models** *of* $\boldsymbol{\lambda}^{\mathbb{A}}_{\to\times}$

Certainly, in a general CCC $\mathcal{C}$ there may be equalities which are *not* reflected by the semantic translation:

$$[\![s]\!] = [\![t]\!] \quad \text{yet} \quad s \neq_{\beta\eta} t$$

We will now *construct* a CCC $\mathcal{C}_\lambda$ in which all equalities between arrows are translations of $\beta\eta$-conversions between terms.

As $\mathcal{C}$ is (very) dependent on syntax (it is called a ***term model***), it is not very useful as a model. It is important, though, from a theoretical perspective as testimony of the existence of a complete model.

Term equivalence

Let us first define the following equivalence relations of *typed* terms. We let:

$$(x, t) \sim_{A,B} (y, u)$$

if $x : A \vdash t : B$ and $y : A \vdash u : B$ are derivable and

$$t =_{\beta\eta} u[x/y]$$

This yields an equivalence relation, so we set:

$$[(x, t)]_{A,B} := \{ \ (y, u) \mid (x, t) \sim_{A,B} (y, u) \ \}$$

Similarly,

$$(\,\blacksquare\,, t) \sim_{\blacksquare,B} (\,\blacksquare\,, u)$$

if $\vdash t : B$ and $\vdash u : B$ are derivable and $t =_{\beta\eta} u$. Moreover,

$$[(\,\blacksquare\,, t)]_{\blacksquare,B} := \{ \ (\,\blacksquare\,, u) \mid (\,\blacksquare\,, t) \sim_{\blacksquare,B} (\,\blacksquare\,, u) \ \}$$

We denote $[(x, t)]_{A,B}$ simply as $[x, t]$.

- Objects:

$$Ob(\mathcal{C}_\lambda) := \{\mathbf{1}\} \cup \{\ \widetilde{A} \mid A \in \textbf{Typ}\ \}$$

- Arrows:

$$\mathcal{C}_\lambda(\widetilde{A}, \widetilde{B}) := \{\ [x, t] \mid x : A \vdash t : B \ \text{ is derivable }\}$$

$$\mathcal{C}_\lambda(\mathbf{1}, \widetilde{B}) := \{\ [\,\centerdot\,, t] \mid\ \vdash t : B \ \text{ is derivable }\}$$

$$\mathcal{C}_\lambda(E, \mathbf{1}) := \{\ \tau_E\ \}$$

- Identities:

$$\mathrm{id}_{\widetilde{A}} := [x, x]\,, \quad \mathrm{id}_{\mathbf{1}} := \tau_{\mathbf{1}}$$

- Composition:

$$[x, t] \circ [y, u] := [y, t[u/x]] \qquad (y \neq x)$$

$$[x, t] \circ [\,\centerdot\,, u] := [\,\centerdot\,, t[u/x]]$$

$$[\,\centerdot\,, t] \circ \tau_{\widetilde{A}} := [y, t] \qquad (y \notin \mathrm{FV}(t))$$

$$[\,\centerdot\,, t] \circ \tau_{\mathbf{1}} := [\,\centerdot\,, t]$$

$$\tau_D \circ h := \tau_E \qquad (h \in \mathcal{C}_\lambda(E, D))$$

Lemma. *$\mathcal{C}_\lambda$ is a category.*

Proof. It is not difficult to see that id's are identities. For associativity, we show the most interesting case (and leave the rest as an exercise):

$$[x, t] \circ ([y, u] \circ [z, v]) = [x, t] \circ [z, u[v/y]] = [z, t[(u[v/y])/x]]$$

$$([x, t] \circ [y, u]) \circ [z, v] = [y, t[u/x]] \circ [z, v] = [z, t[u/x][v/y]]$$

Since $y \not\equiv x$ and t has at most x as a free variable, y is not free in t and therefore (Exercise 3 from Lecture 1):

$$t[u/x][v/y] \equiv t[(u[v/y])/x]$$

□

Lemma. *$\mathcal{C}_\lambda$ has finite products.*

Proof. Clearly, $\mathbf{1}$ is terminal with canonical arrows $\tau_E : E \longrightarrow \mathbf{1}$.

For (binary) products, $\mathbf{1} \times E = E \times \mathbf{1} := E$. Otherwise, define $\widetilde{A} \xleftarrow{\pi_1} \widetilde{A} \times \widetilde{B} \xrightarrow{\pi_2} \widetilde{B}$:

$$\widetilde{A} \times \widetilde{B} := \widetilde{A \times B}$$
$$\pi_i := [x, \pi_i x] \quad i = 1, 2$$

For $\widetilde{A} \xleftarrow{[x,t]} \widetilde{C} \xrightarrow{[x,u]} \widetilde{B}$, take $\langle [x,t],[x,u] \rangle : \widetilde{C} \longrightarrow \widetilde{A} \times \widetilde{B} := [x, \langle t, u \rangle]$. Then,

$$\begin{aligned} \pi_1 \circ \langle [x,t],[x,u] \rangle &= [y, \pi_1 y] \circ [x, \langle t, u \rangle] && \text{Definitions} \\ &= [x, \pi_1 \langle t, u \rangle] && \text{Defn of composition} \\ &= [x, t] && \beta\text{-conversion} \end{aligned}$$

Uniqueness (η) left as an exercise. $\square$

Lemma. $\mathcal{C}_\lambda$ *has exponentials.*

Proof. We have that $\mathbf{1} \Rightarrow E := E$ and $E \Rightarrow \mathbf{1} := \mathbf{1}$, with obvious evaluation arrows. Otherwise,

$$\widetilde{B} \Rightarrow \widetilde{A} := \widetilde{B \to A}$$

$$\mathsf{app} : (\widetilde{B} \Rightarrow \widetilde{A}) \times \widetilde{B} \longrightarrow \widetilde{A} := [x, (\pi_1 x)(\pi_2 x)]$$

For $[x, t] : \widetilde{C} \times \widetilde{B} \longrightarrow \widetilde{A}$, take $\Lambda([x, t]) := [x_1, \lambda x_2.t[\langle x_1, x_2\rangle/x]]$.

So,
$$\begin{aligned}
\mathsf{app} \circ (\Lambda([x,t]) \times \mathsf{id}) &= \mathsf{app} \circ \langle \Lambda([x,t]) \circ \pi_1, \mathsf{id} \circ \pi_2\rangle \\
&= \mathsf{app} \circ \langle [x_1, \lambda x_2.t[\langle x_1, x_2\rangle/x]] \circ [y, \pi_1 y], [y, \pi_2 y]\rangle \\
&= \mathsf{app} \circ \langle [y, \lambda x_2.t[\langle \pi_1 y, x_2\rangle/x]], [y, \pi_2 y]\rangle \\
&= [z, (\pi_1 z)(\pi_2 z)] \circ [y, \underbrace{\langle \lambda x_2.t[\langle \pi_1 y, x_2\rangle/x], \pi_2 y\rangle}_{u}] \\
&= [y, (\pi_1 u)(\pi_2 u)] \stackrel{\beta}{=} [y, (\lambda x_2.t[\langle \pi_1 y, x_2\rangle/x])(\pi_2 y)] \\
&\stackrel{\beta}{=} [y, t[\langle \pi_1 y, \pi_2 y\rangle/x]] \stackrel{\eta}{=} [y, t[y/x]] = [x, t]
\end{aligned}$$

Uniqueness (η) and other cases left as exercise. □

Applying our translation from the λ-calculus to a CCC we have

$$[\![\Gamma \vdash t : A]\!] = [x,\ t[\pi_i x / x_i]_{i=1..n}]$$

where $\Gamma = \{x_1 : A_1, ..., x_n : A_n\}$, $x \notin \Gamma$ and $x : \prod_{i=1..n} A_i$.

Then,

$$\begin{aligned}
&[\![\Gamma \vdash t : A]\!] = [\![\Gamma \vdash u : A]\!] \implies [x,\ t[\pi_i x/x_i]_i] = [x,\ u[\pi_i x/x_i]_i] \\
&\implies t[\pi_i x/x_i]_i =_{\beta\eta} u[\pi_i x/x_i]_i \\
&\implies (\lambda x.t[\pi_i x/x_i]_i)\langle x_1, ..., x_n\rangle =_{\beta\eta} (\lambda x.u[\pi_i x/x_i]_i)\langle x_1, ..., x_n\rangle \\
&\implies t[\pi_i\langle x_1, ..., x_n\rangle/x_i]_i =_{\beta\eta} u[\pi_i\langle x_1, ..., x_n\rangle/x_i]_i \\
&\implies t[x_i/x_i]_i =_{\beta\eta} u[x_i/x_i]_i \implies t =_{\beta\eta} u
\end{aligned}$$

and, using also soundness of the CCC translation:

$$[\![\Gamma \vdash t : A]\!] = [\![\Gamma \vdash u : A]\!] \iff t =_{\beta\eta} u$$

Wrapping up

- *We started this journey with a minimal language for functions, the untyped λ-calculus, which we proved to be Turing powerful.*
- *We examined disciplined restrictions using types.*
- *We looked at a correspondence at the Foundations of Computer Science.*

The End

Exercises

1. Give Natural Deduction proofs of the following sequents.
   (a) $\vdash (A \supset B) \supset ((B \supset C) \supset (A \supset C))$
   (b) $\vdash (A \supset (A \supset B)) \supset (A \supset B)$
   (c) $\vdash (C \supset A) \supset ((C \supset B) \supset (C \supset (A \wedge B)))$
   (d) $\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

   In each case, give a λ-term for the corresponding proof.
2. We have seen that each numeral $\ulcorner n \urcorner$ can be typed with $(A \to A) \to A \to A$. What does this tell us about the number of proofs in normal form of the formula $(A \supset A) \supset A \supset A$?
3. Complete the proofs of the lemmata on slides 22, 23 and 24.
4. This exercise guides you through a proof of weak normalisation of λ-terms typable in Simple Types has a normal form. Moreover, we will derive a bound on the size of the normal form. (The bound is huge, but necessarily so.)
   We consider the simply-typed λ-calculus with Church types (see Lecture 4, slides 25-26). Recall that terms are given by the rules:
   - if $x^A \in \mathcal{V}^A$ then $x^A \in \Lambda^A$,
   - if $s \in \Lambda^{A \to B}$ and $t \in \Lambda^A$ then $st \in \Lambda^B$,
   - if $s \in \Lambda^B$ and $x^A \in \mathcal{V}^A$ then $\lambda x^A . s \in \Lambda^{A \to B}$.

   Note that, in this case, the type of each term can be deduced unambiguously. For example, $\lambda x^a . x$ can only have type $a \to a$.

Now let us define the *height* of a term by:

- $\mathsf{h}(x) = 1$,
- $\mathsf{h}(\lambda x : T.\, t) = \mathsf{h}(t) + 1$,
- $\mathsf{h}(tu) = \max(\mathsf{h}(t), \mathsf{h}(u)) + 1$.

Consider also the notion of *rank* of a type, $\rho(T)$:

- the rank of an atomic type is 0,
- while $\rho(T \to U) = \max(\rho(T) + 1, \rho(U))$.

The rank of a term of type A is defined to be the rank of A, while the *redex-rank* of a redex $(\lambda x^A .\, t)u$ is defined to be the rank of $\lambda x^A .\, t$. The *degree* of a term, $\mathsf{d}(t)$, is defined to be the maximum redex-rank of any redex occuring in t, or 0 if t is a normal form.

(a) Show that for any terms t, u, if $t \to_\beta u$ then $\mathsf{d}(u) \leq \mathsf{d}(t)$. Use induction on $\mathsf{h}(t)$.

(b) Suppose we have $u : B$ and $t : A$, while $x^A \in \mathcal{V}^A$. Show that $\mathsf{h}(u[t/x]) \leq \mathsf{h}(t) + \mathsf{h}(u)$.

(c) Prove that any term t such that $\mathsf{d}(t) \leq d \geq 1$ and $\mathsf{h}(t) \leq h$ can be β-reduced (in finitely many reduction steps) to a term u of degree $\leq d - 1$ and height $\leq 2^h$. Use induction on $\mathsf{h}(t)$.

(d) Now define $e(m, n)$ by $e(m, 0) = m$, $e(m, n+1) = 2^{e(m,n)}$. Thus $e(m, n)$ is an exponential 'stack' of n 2's with an m at the top. Prove that a term of degree d and height h has a normal form of height bounded by $e(h, d)$. Use the previous part, and induction on the degree.