Lambda Calculus and Types

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Lecture 2: Normal Forms and Consistency

The Story So Far

- Terms, free variables, closed terms.

(var) $\dfrac{}{x \in \Lambda}\, x \in \mathcal{V}$ (app) $\dfrac{s \in \Lambda \quad t \in \Lambda}{(st) \in \Lambda}$ (abs) $\dfrac{s \in \Lambda}{(\lambda x.s) \in \Lambda}\, x \in \mathcal{V}$

- Alpha conversion; the Variable Convention.

> $s \equiv t$ if one can derive the same term from both purely by renaming bound variables to **fresh variables**.

- Notion of reduction β and capture-avoiding substitution.

$$\beta = \{\, \langle (\lambda x.s)t, s[t/x] \rangle \mid s, t \in \Lambda,\ x \in \mathcal{V} \,\}$$

- Example combinators (i.e. closed terms): $\mathbf{i} := \lambda x.x$, $\mathbf{k} := \lambda xy.x$, $\mathbf{\Omega} := (\lambda x.xx)(\lambda x.xx)$, $\mathbf{y} := \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$, etc.

Notions of Reduction

A ***notion of reduction*** over Λ, also called a ***redex rule***, is just a binary relation on Λ, i.e. a subset of $\Lambda \times \Lambda$.

From a notion of reduction R we can derive the relations on $\Lambda \times \Lambda$:

- The ***one-step R-reduction*** is the relation $\to_R$ defined by:

$$\frac{}{s \to_R t}\ \langle s, t\rangle \in R \qquad \frac{s \to_R t}{su \to_R tu} \qquad \frac{s \to_R t}{us \to_R ut} \qquad \frac{s \to_R t}{\lambda x.s \to_R \lambda x.t}$$

  We read $s \to_R t$ as "s R-reduces to t in one step"

- $\to_R^=$ is the reflexive closure of $\to_R$
- $\to_R^+$ is the transitive closure of $\to_R$
- $\twoheadrightarrow_R$ is the reflexive, transitive closure of $\to_R$
  Read $s \twoheadrightarrow_R t$ as "s R-reduces to t"
- $=_R$ is the reflexive, symmetric, transitive closure of $\to_R$
  Read $s =_R t$ as "s R-converts to t" (or "s is R-convertible to t")

Notions of Reduction (rules)

(R)	$\dfrac{}{s \to_R^= t}\ \langle s,t\rangle \in R$	$\dfrac{}{s \to_R^+ t}\ \langle s,t\rangle \in R$	$\dfrac{}{s \twoheadrightarrow_R t}\ \langle s,t\rangle \in R$	$\dfrac{}{s =_R t}\ \langle s,t\rangle \in R$
(l-app)	$\dfrac{s \to_R^= t}{su \to_R^= tu}$	$\dfrac{s \to_R^+ t}{su \to_R^+ tu}$	$\dfrac{s \twoheadrightarrow_R t}{su \twoheadrightarrow_R tu}$	$\dfrac{s =_R t}{su =_R tu}$
(r-app)	$\dfrac{s \to_R^= t}{us \to_R^= ut}$	$\dfrac{s \to_R^+ t}{us \to_R^+ ut}$	$\dfrac{s \twoheadrightarrow_R t}{us \twoheadrightarrow_R ut}$	$\dfrac{s =_R t}{us =_R ut}$
(abs)	$\dfrac{s \to_R^= t}{\lambda x.s \to_R^= \lambda x.t}$	$\dfrac{s \to_R^+ t}{\lambda x.s \to_R^+ \lambda x.t}$	$\dfrac{s \twoheadrightarrow_R t}{\lambda x.s \twoheadrightarrow_R \lambda x.t}$	$\dfrac{s =_R t}{\lambda x.s =_R \lambda x.t}$
(refl)	$\dfrac{}{s \to_R^= s}$		$\dfrac{}{s \twoheadrightarrow_R s}$	$\dfrac{}{s =_R s}$
(trans)		$\dfrac{s \to_R^+ u \quad u \to_R^+ t}{s \to_R^+ t}$	$\dfrac{s \twoheadrightarrow_R u \quad u \twoheadrightarrow_R t}{s \twoheadrightarrow_R t}$	$\dfrac{s =_R u \quad u =_R t}{s =_R t}$
(sym)				$\dfrac{t =_R s}{s =_R t}$

Lemma. *Let R be a notion of reduction over Λ and let s, t be terms.*

- $s \to_R^+ t$ *if and only if for some* $n \geq 1$ *and terms* $s_0, s_1, \ldots, s_n$,

$$s \equiv s_0 \to_R s_1 \to_R \cdots \to_R s_n \equiv t$$

- $s \twoheadrightarrow_R t$ *if and only if for some* $n \geq 0$ *and terms* $s_0, s_1, \ldots, s_n$,

$$s \equiv s_0 \to_R s_1 \to_R \cdots \to_R s_n \equiv t$$

- $s =_R t$ *if and only if for some* $n \geq 0$ *and* $s_0, s_1, \ldots, s_n, t_0, t_1, \ldots, t_n$,

$$\begin{tikzcd}
 & t_0 \arrow[dl, two heads, "R"'] \arrow[dr, two heads, "R"] & & t_1 \arrow[dl, two heads, "R"'] & \cdots & t_{n-1} \arrow[dr, two heads, "R"] & & t_n \equiv t \arrow[dl, two heads, "R"'] \\
s \equiv s_0 & & s_1 & & \cdots & & s_n &
\end{tikzcd}$$

- one-step β-reduction:

$$\frac{}{(\lambda x.s)t \to_\beta s[t/x]} \quad \frac{s \to_\beta t}{su \to_\beta tu} \quad \frac{s \to_\beta t}{us \to_\beta ut} \quad \frac{s \to_\beta t}{\lambda x.s \to_\beta \lambda x.t}$$

  from which we obtain $\to_\beta^=$, $\to_\beta^+$, $=_\beta$. For example:

  - $(\lambda x.x)(\lambda yz.z) \to_\beta \lambda yz.z$
  - $\lambda x.x((\lambda y.y)x) \to_\beta \lambda x.xx$
  - $(\lambda z.zz)((\lambda x.x)y) \twoheadrightarrow_\beta yy \ {}_\beta\!\twoheadleftarrow (\lambda z.z)((\lambda x.xx)y)$

    so $(\lambda z.zz)((\lambda x.x)y) =_\beta (\lambda z.z)((\lambda x.xx)y)$
  - recall $\mathbf{\Omega} := (\lambda x.xx)(\lambda x.xx)$, thus $\mathbf{\Omega} \to_\beta \mathbf{\Omega} \to_\beta \mathbf{\Omega} \to_\beta \mathbf{\Omega} \to_\beta \cdots$
- Fixed point combinators, First Recursion Theorem.
  - A closed term u is said to be a ***fixed point combinator*** if, for all s:

Fixed point combinators are e.g: $\mathbf{y}, \boldsymbol{\theta}, \mathtt{mgs}, \ldots$

$$s(us) =_\beta us$$

Normal Forms

Let R be a notion of reduction.

- A term s is ***in R-normal form*** if there is no term t such that $s \to_R t$.
- A term s ***has a normal form*** (or *is normalizable* (w.r.t. $\to_R$)) if there is a term t in normal form with $s \twoheadrightarrow_R t$. We say that t *is a normal form for* s.
- A term s is ***strongly normalizable*** (w.r.t. $\to_R$) if there is no infinite sequence $t_1, t_2, \ldots$ such that:

$$s \to_R t_1 \to_R t_2 \to_R t_3 \to_R \ldots$$

  (i.e. every sequence of one-step R-reductions from s eventually reaches a nf.)
- $\to_R$ is ***weakly normalizing*** if every term is normalizable.
- $\to_R$ is ***strongly normalizing*** if every term is strongly normalizable.

s strongly normalizable	$\Longrightarrow$	s normalizable
$\to_R$ strongly normalizing	$\Longrightarrow$	$\to_R$ weakly normalizing

The Diamond and Church-Rosser Properties

Let $\to_R$ be a binary relation and $\twoheadrightarrow_R$ its reflexive transitive closure.

- $\to_R$ satisfies the ***diamond property*** if, for all terms s, t_1, t_2,

$$\begin{tikzcd} & s \arrow[dl, "R"'] \arrow[dr, "R"] & \\ t_1 & & t_2 \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} t_1 \arrow[dr, "R"'] & & t_2 \arrow[dl, "R"] \\ & u & \end{tikzcd} \quad \text{for some } u.$$

- $\to_R$ is ***Church-Rosser*** if $\twoheadrightarrow_R$ satisfies the diamond property.

Lemma. *If $\to_R$ satisfies the diamond property then it is Church-Rosser.*

Properties of β-reduction

- What kind of terms are in normal form?
- Do all terms have a normal form?
- Is β-reduction weakly/strongly normalising?
- Does it satisfy the diamond or Church-Rosser properties?

In order to show that $\to_\beta$ is Church-Rosser we follow the method below.

- We define an auxiliary relation $\to_{||}$ such that:

  (a) $\to_\beta \subseteq \to_{||} \subseteq \twoheadrightarrow_\beta$

  (b) and, moreover, it satisfies the diamond property.

- By taking reflexive transitive closures in (a) we obtain $\twoheadrightarrow_{||} = \twoheadrightarrow_\beta$.
- By previous lemma and (b), $\to_{||}$ is CR.

Parallel β-reduction

Definition. ***Parallel β-reduction***, written $\to_{\parallel}$, is the binary relation on terms defined by:

$$(\text{refl})\ \frac{}{s \to_{\parallel} s} \qquad (\parallel\text{-app})\ \frac{s \to_{\parallel} s' \quad t \to_{\parallel} t'}{st \to_{\parallel} s't'}$$

$$(\text{abs})\ \frac{s \to_{\parallel} s'}{\lambda x.s \to_{\parallel} \lambda x.s'} \qquad (\parallel\text{-}\beta)\ \frac{s \to_{\parallel} s' \quad t \to_{\parallel} t'}{(\lambda x.s)t \to_{\parallel} s'[t'/x]}$$

Lemma. *If $s \to_{\parallel} s'$ and $t \to_{\parallel} t'$ then $s[t/x] \to_{\parallel} s'[t'/x]$.*

Lemma. *$\to_{\parallel}$ satisfies the diamond property.*

Lemma. *$\to_{\beta} \subseteq \to_{\parallel} \subseteq \twoheadrightarrow_{\beta}$ and therefore $\twoheadrightarrow_{\beta}$ is the reflexive transitive closure of $\to_{\parallel}$.*

Theorem. *$\to_{\beta}$ is Church-Rosser.*

We say that $\to_R$ has the ***unique normal form*** property if, whenever

$$\begin{tikzcd} & s \arrow[dl, two heads, "R"'] \arrow[dr, two heads, "R"] & \\ t_1 & & t_2 \end{tikzcd}$$

with t_1, t_2 in normal form, we have that $t_1 \equiv t_2$.

Lemma. *If $\to_R$ is Church-Rosser then:*

- $s =_R t$ *if and only if there is some* u *with* $s \twoheadrightarrow_R u$ *and* $t \twoheadrightarrow_R u$.
- $\to_R$ *has the unique normal form property.*

Lemma. *β-conversion is consistent, that is, there exist terms t_1, t_2 such that $t_1 \neq_\beta t_2$.*

Proof. Suppose, say, $\mathbf{i} =_\beta \mathbf{k}$. Then, by C-R and previous lemma, $\mathbf{i} \twoheadrightarrow_\beta u$ and $\mathbf{k} \twoheadrightarrow_\beta u$, for some u, contradiction as $\mathbf{i}, \mathbf{k}$ are normal forms. $\square$

There is more in functions than beta

– *Are there any meaningful ways to extend β (and remain consistent)?*

Here is an *extensional viewpoint* on functions:

two functions are equal if they yield the same output on each input

leading to the following notion of reduction.

$$\eta = \{ \langle \lambda x.sx,\ s \rangle \mid s \in \Lambda, x \notin \mathsf{FV}(s) \}$$

We write $\beta\eta$ for $\beta \cup \eta$.

For all terms s, t and any variable x not free in st:

$$sx =_{\beta\eta} tx \implies \lambda x.sx =_{\beta\eta} \lambda x.tx \implies s =_{\beta\eta} t$$

Theorem. *$\to_{\beta\eta}$ satisfies the Church-Rosser property, hence $\beta\eta$-conversion is consistent.*

(There is also η-expansion: $\eta^{\exp} = \{ \langle s,\ \lambda x.sx \rangle \mid s \in \Lambda, x \notin \mathsf{FV}(s) \}$.)

Is there more than beta-eta?

– *Are there any other meaningful ways to extend β (and remain consistent)?*

Not that many. Define two standard combinators:

$$\mathbf{t} := \lambda xy.x \ (\equiv \mathbf{k}) \qquad \mathbf{f} := \lambda xy.y$$

Theorem (Böhm)**.** *Let s and t be distinct closed $\beta\eta$-normal forms. Then there exist terms $u_1, \ldots, u_n$ such that*

$$su_1u_2 \ldots u_n =_\beta \mathbf{t} \quad \textit{and} \quad tu_1u_2 \ldots u_n =_\beta \mathbf{f}.$$

Hence, if s and t are distinct closed $\beta\eta$-normal forms then any $=_R$ containing $=_\beta$ and equating s and t is inconsistent.
In particular, $s \neq_\beta t$ (which we knew already by C-R of $\to_\beta$).

Non-termination?

– *Is it consistent to extend β by equating all terms without normal forms?*

No. For example, equating $x\mathbf{\Omega}$ with $\mathbf{\Omega}$ leads to inconsistency.

Summary of consistency results:

- $=_\beta$ and $=_{\beta\eta}$ are consistent.
- $=_\beta$ does not equate distinct β-normal forms;
  $=_{\beta\eta}$ does not equate distinct $\beta\eta$-normal forms.
- It is inconsistent to extend β by equating any terms with distinct $\beta\eta$-normal forms.
- It is inconsistent to extend β by equating all terms without a normal form.

Reduction Strategies

Definition. A ***reduction strategy*** is a binary relation $\to$ satisfying:

$$\forall s, t_1, t_2. \left(s \to t_1 \wedge s \to t_2 \implies t_1 \equiv t_2 \right)$$

- A reduction strategy can also be thought of as a (partial) function, mapping each term to a unique reduct, if it has a reduct at all.
- Reduction strategies automatically have the unique normal form property.
- Usually reduction strategies are **not** compatible with the term formation rules.

Leftmost Reduction

An important strategy is:

- ***Leftmost Reduction***:

  $s \xrightarrow{l} t$ if $s \to_\beta t$ and the redex $(\lambda x.p)q$ reduced in s is the leftmost amongst all β-redexes in s (measuring the position of a redex by its initial λ).

Theorem. *If $s \twoheadrightarrow_\beta t$, where t is a β-normal form, then $s \xrightarrow{l}\!\!\!\twoheadrightarrow t$. Put otherwise, leftmost reduction is sufficient to find the normal form of any normalising term.*

- A very important result for the implementation of functional programming languages.
- Also useful for showing that some terms have no normal form. E.g. $\mathbf{y}(\lambda x.x)$ does not have a normal form, and $\mathbf{\Omega} s$ does not have a normal form for any s.

Computability: Booleans

The (untyped) λ-calculus is **so** expressive that we can encode in it (computations with) Booleans, Numerals, etc.

Let us define (again):

$$\mathbf{t} := \lambda xy.x$$
$$\mathbf{f} := \lambda xy.y$$

These correspond to the booleans "true" and "false". For example, setting

$$\textbf{and} := \lambda xy.xy\mathbf{f}$$

we obtain:

$$\textbf{and}\,\mathbf{f}\,\mathbf{f} \twoheadrightarrow_\beta \mathbf{f} \qquad \textbf{and}\,\mathbf{f}\,\mathbf{t} \twoheadrightarrow_\beta \mathbf{f}$$
$$\textbf{and}\,\mathbf{t}\,\mathbf{f} \twoheadrightarrow_\beta \mathbf{f} \qquad \textbf{and}\,\mathbf{t}\,\mathbf{t} \twoheadrightarrow_\beta \mathbf{t}$$

We can also define a *conditional* (if-then-else) combinator:

$$\textbf{cond} := \lambda x.x$$

Natural Numbers

The number n is encoded by the ***numeral***:

$$\ulcorner n \urcorner := \lambda f x.\, \underbrace{f(f(f \ldots (f}_{n\text{-fold app}}\, x))$$

This is Church's original way to encode numbers in the untyped λ-calculus.

For example:

$$\begin{aligned}
\ulcorner 0 \urcorner &:= \lambda f x.x \quad (\equiv \mathbf{f}) \\
\ulcorner 1 \urcorner &:= \lambda f x.fx \\
\ulcorner 2 \urcorner &:= \lambda f x.f(fx) \\
\ulcorner 3 \urcorner &:= \lambda f x.f(f(fx)) \\
&\vdots
\end{aligned}$$

E.g. what does the combinator $\lambda n f x.f(nfx)$ do on numerals?

Combinators for Numerals

- **succ** $:= \lambda nfx.f(nfx)$ satisfies **succ**$\ulcorner n \urcorner =_\beta \ulcorner n+1 \urcorner$
- **zero?** $:= \lambda n.n(\lambda x.\mathbf{f})\mathbf{t}$ satisfies $\begin{cases} \textbf{zero?}\ulcorner 0 \urcorner =_\beta \mathbf{t} \\ \textbf{zero?}\ulcorner n+1 \urcorner =_\beta \mathbf{f} \end{cases}$
- **add** $:= \lambda mnfx.mf(nfx)$ satisfies **add**$\ulcorner m \urcorner \ulcorner n \urcorner =_\beta \ulcorner m+n \urcorner$
- **mult** $:= \lambda mnf.m(nf)$ satisfies **mult**$\ulcorner m \urcorner \ulcorner n \urcorner =_\beta \ulcorner m \times n \urcorner$
- **pred′** $:= \lambda n.n(\lambda y.y\mathbf{i}(\textbf{succ}\, y))(\lambda ab.\ulcorner 0 \urcorner)$ sat. **pred′**$\ulcorner n+1 \urcorner =_\beta \ulcorner n \urcorner$
- We can ‘program’ a combinator **pred** such that **pred** $\ulcorner 0 \urcorner =_\beta \ulcorner 0 \urcorner$ and **pred** $\ulcorner n+1 \urcorner =_\beta \ulcorner n \urcorner$.
- **rcase** $:= \lambda nfg.$ **zero?**$nf(g(\textbf{pred}\, n))$ satisfies

$$\textbf{rcase}\ulcorner n \urcorner fg =_\beta \begin{cases} f & \text{, if } n = 0 \\ g(\ulcorner n-1 \urcorner) & \text{, if } n > 0 \end{cases}$$

Exercises

Below we use the combinators $\mathbf{i} := \lambda x.x$, $\mathbf{k} := \lambda xy.x$, $\mathbf{s} := \lambda xyz.xz(yz)$ and $\mathbf{y} := \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.

1. Give two terms which are:
   (a) in β-normal form.
   (b) not in β-normal form but strongly normalizable.
   (c) normalizable but not strongly normalizable.
   (d) not normalizable.
2. Verify that $\rightarrow_\beta$ does not satisfy the diamond property by specifically considering one-step β-reductions from the following terms.
   (a) $(\lambda x.\mathbf{i})(\mathbf{ii})$
   (b) $(\lambda x.xx)(\mathbf{ii})$

   Observe how the number of occurrences of x disallows us to close the diamonds.
3. Prove the theorem on slide 10 (using the lemmas above it).
4. Prove the first lemma on slide 11.
5. Without using Böhm's Theorem, show that for any $=_\beta \subseteq =_R$:
   (a) If $\mathbf{i} =_R \mathbf{k}$ then $=_R$ is inconsistent.
   (b) If $\mathbf{i} =_R \mathbf{s}$ then $=_R$ is inconsistent.
   (c) If $xy = yx$, for some variables x and y, then $=_R$ is inconsistent.
6. Which of the following terms have β-normal forms?
   (a) $\mathbf{sk}$.
   (b) $(\lambda x.xxx)(\lambda x.xxx)$.
   (c) $\mathbf{yi}$.
   (d) $\mathbf{y}(\lambda yz.zy)$.

   [Hint: You just need to give the leftmost reduction sequence.]

The next exercises extend the proof of the Church-Rosser property to $\beta\eta$-reduction.
Let $\rightarrow_1$ and $\rightarrow_2$ be binary relations over Λ. We say that $\rightarrow_1$ and $\rightarrow_2$ commute if $s \rightarrow_1 t_1$ and $s \rightarrow_2 t_2$ implies that there exists t with $t_1 \rightarrow_2 t$ and $t_2 \rightarrow_1 t$.

7. Let $\rightarrow_1$ and $\rightarrow_2$ be binary relations over Λ which commute, and suppose that both satisfy the diamond property. Define $\rightarrow_{12}$ to be the union of the two relations, and $\twoheadrightarrow_{12}$ the transitive closure of $\rightarrow_{12}$. Show that $\twoheadrightarrow_{12}$ satisfies the diamond property.
8. Let $\rightarrow_\eta$ be the one-step η-reduction. Show that $\twoheadrightarrow_\eta$ and $\twoheadrightarrow_\beta$ commute.
   [Hint: It suffices to show (by case analysis on possible redexes) that if $s \rightarrow_\beta s_1$ and $s \rightarrow_\eta s_2$ then there is t such that $s_1 \twoheadrightarrow_\eta t$ and $s_2 \rightarrow_\beta^= t$, then do a diagram chase.]
9. Show that $\rightarrow_\eta$ is Church-Rosser.
   [Hint: It suffices to show (by case analysis on possible redexes) that if $s \rightarrow_\eta s_1$ and $s \rightarrow_\eta s_2$ then there is t such that $s_1 \rightarrow_\eta^= t$ and $s_2 \rightarrow_\eta^= t$, then do a diagram chase.]
10. Use the above results to show that $\rightarrow_{\beta\eta}$ is Church-Rosser.